International Journal of Modern Research in Engineering and Technology (IJMRET)
www.ijmret.org Volume 1 Issue 5 ǁ December 2016.

The Bifurcation of Stage Structured Prey-Predator Food Chain Model with Refuge

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ABSTRACT *:In this paper, we established the condition of the occurrence of local bifurcation (such as saddle-node, transcritical and pitchfork)with particular emphasis on the hopf bifurcation near of the positive equilibrium point of ecological mathematical model consisting of prey-predator model involving prey refuge with two different function response are established. After the study and analysis, of the observed incidence transcritical bifurcation near equilibrium point E_0, E_1, E_2 as well as the occurrence of saddle-node bifurcation at equilibrium point E_3.It is worth mentioning, there are no possibility occurrence of the pitch fork bifurcation at each point. Finally, some numerical simulation are used to illustration the occurrence of local bifurcation o this model.*

KEYWORDS *-ecological model, Equilibrium point, Local bifurcation, Hopf bifurcation.*

I. INTRODUCTION

A bifurcation mean there is a change in the stability of equilibria of the system at the value. Many, if not most differential equations depend of parameters. Depending on the value of these parameters, the qualitative behavior of systems solution can be quite different [1].

Bifurcation theory studies the qualitative changes in the phase portrait, for example, the appearance and disappearance of equilibria, periodic orbits, or more complicated features such as strange attractors. The methods and results of bifurcation theory are fundamental to an understanding of nonlinear dynamical systems. The bifurcationis divided into two principal classes: local bifurcations and global bifurcations. Local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbit or other invariant sets as parameters cross through critical thresholds such as saddle node, transcritical, pitchfork, period-doubling (flip), Hopf and Neimark (secondary Hopf) bifurcation. Global bifurcations occur when larger invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the trajectories in phase space which cannot be confined to a small neighborhood, as is the case with local bifurcations. In fact, the changes in topology extend out of an arbitrarily large distance (hence “global”). Such as homoclinic in which a limit cycle collides with a saddle point, and heteroclinic bifurcation in which a limit cycle collides with two or more saddle points, and infinite-periodic bifurcation in which a stable node and saddle point simultaneously occur on a limit cycle [2].

the name "bifurcation" in 1885 in the first paper in mathematics showing such a behavior also later named various types of stationary points and classified them. Perko L. [3] established the conditions of the occurrence of local bifurcation (such as saddle-node, transcritical and pitchfork). However,the necessary condition for the occurrence of the Hopf bifurcation presented by Hirsch M.W. and Smale S. [4].

In this chapter, we will establish the condition of the occurrence of local bifurcation and Hopf bifurcation around each of the equilibrium point of a mathematical model proposed by AzharM,and Noor Hassan [5].

II. Model formulation [5]:

Anecological mathematical model consisting of prey-predator model involving prey refuge with two different function response is proposed and analyzed in [5].

$$\frac{dW_1}{dT} = \alpha\, W_2\left(1 - \frac{W_2}{k}\right) - \beta\, W_1$$
$$\frac{dW_2}{dT} = \beta\, W_1 - d_1 W_2 - c_1(1-m)W_2 \quad (1)$$
$$\frac{dW_3}{dT} = e_1 c_1(1-m)W_2\, W_3 - d_2 W_3 - \frac{c_2 W_3}{b + W_3}\, W_4$$
$$\frac{dW_4}{dT} = \frac{e_2 c_2 W_3}{b + W_3}\, W_4 - d_3 W_4$$

with initial conditions $W_i(0) \geq 0,\ i = 1, 2, 3, 4$. Not that the above proposed model has twelve parameters in all which make the analysis difficult. So in order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters :

$$t = \alpha\, T\ , a_1 = \frac{\beta}{\alpha}\ , a_2 = \frac{d_1}{\alpha}\ , a_3 = \frac{d_2}{\alpha}\ ,$$
$$a_4 = \frac{e_1 c_1\, k}{\alpha}, a_5 = \frac{c_2}{\alpha}\ , a_6 = \frac{b\, c_1}{\alpha},$$
$$a_7 = \frac{e_2 c_2}{\alpha}\ , a_8 = \frac{d_3}{\alpha}\ , w_1 = \frac{W_1}{k}\ , w_2 = \frac{W_2}{k}\ ,$$
$$w_3 = \frac{c_1 W_3}{\alpha}\ , w_4 = \frac{c_1 W_4}{\alpha}\ .$$

Then the non-dimensional from of system (1) can be written as:

$$\frac{dw_1}{dt} = w_1\left[\frac{w_2(1-w_2)}{w_1} - a_1\right]$$
$$= w_1 f_1(w_1, w_2, w_3, w_4)$$

$$\frac{dw_2}{dt} = w_2\left[\frac{a_1 w_1}{w_2} - a_2 - (1-m)w_3\right]$$
$$= w_2 f_2(w_1, w_2, w_3, w_4) \qquad (2)$$
$$\frac{dw_3}{dt} = w_3\left[a_4(1-m)w_2 - a_3 - \frac{a_5 w_4}{a_6 + w_3}\right]$$
$$= w_3 f_3(w_1, w_2, w_3, w_4)$$
$$\frac{dw_4}{dt} = w_4\left[\frac{a_7 w_3}{a_6 + w_3} - a_8\right] = w_4 f_4(w_1, w_2, w_3, w_4)$$

with $w_1(0) \geq 0, w_2(0) \geq 0, w_3(0) \geq 0$ and $w_4(0) \geq 0$. It is observed that the number of parameters have been reduced from twelve in the system (1) to nine in the system (2). Obviously the interaction functions of the system (2) are continuous and have continuous partial derivatives on the following positive four dimensional space.

$$R_+^4 = \left\{\begin{matrix}(w_1, w_2, w_3, w_4) \in R^4 : w_1(0) \geq 0, \\ w_2(0) \geq 0, w_3(0) \geq 0 \, w_4(0) \geq 0.\end{matrix}\right\}$$

Therefore these functions are Lipschitzian on R_+^4, and hence the solution of the system(2)exists and is unique. Further, all the solutions of system(2) with non-negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem 1 : All the solutions of system (2) which initiate in R_+^4 are uniformly bounded.

III. The stability Analysis of Equilibrium Points of system (2) [5]

It is observed that, system (2) has at most four biological feasible equilibrium point which are mentioned with their existence condition in [5] as in the following:

1.the Vanishing Equilibrium point: $E_0 = (0,0,0,0)$ always exists and E_0 is locally asymptotically stable in Int R_+^4 if the following condition hold

$$a_2 > 1 \qquad (3.a)$$

However, it is (a saddle point) unstable otherwise. More details see [5]

2.The freepredators equilibrium point $\boldsymbol{E_1 = (\bar{w}_1, \bar{w}_2, 0, 0)}$ *exists uniquely in* $Int. R_+^2$ *if and only if the following condition hold:*

$$\bar{w}_2 < \frac{a_3}{a_4(1-m)} \qquad (3.b)$$

The following second order polynomial equation

$\lambda^2 + A_1\lambda + A_2 = 0$,

which gives the other two eigenvalues of $J(E_0)$ by:

$$\lambda_{0w_1} = \frac{-A_1}{2} + \frac{1}{2}\sqrt{{A_1}^2 - 4A_2} \; ,$$
$$\lambda_{0w_2} = \frac{-A_1}{2} - \frac{1}{2}\sqrt{{A_1}^2 - 4A_2} \; .$$

E_0 is locally asymptotically stable in the R_+^4. However, it is a saddle point otherwise .

3.The freetoppredator equilibrium point $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ *exists uniquely in the* $Int. R_+^3$ *if the following condition are holds:*

$$a_3 < \min\{a_4(1-m),\ a_4(1-m)(1-a_2)\}. \qquad (3\,d)$$

The following three order polynomial equation

$$[\lambda^3 + R_1\lambda^2 + R_2\lambda + R_3]\left(-a_8 + \frac{a_7\breve{w}_3}{a_6 + \breve{w}_3} - \lambda\right) = 0$$

where

$R_1 = -(b_{11} + b_{22}) > 0$,
$R_2 = -(b_{11}b_{22} + b_{23}b_{32} + b_{12}b_{21})$,
$R_3 = b_{11}b_{23}b_{32} > 0$,

So, either $-a_8 + \frac{a_7\breve{w}_3}{a_6+\breve{w}_3} - \lambda = 0$.

Or $\lambda^3 + R_1\lambda^2 + R_2\lambda + R_3 = 0$.

Hence from equation (2.9 c) we obtain that:

$$\lambda_{2w_3} = -a_8 + \frac{a_7\breve{w}_3}{a_6 + \breve{w}_3} \; ,$$

which is negative if inaddition of condition (2. 5 h)in [ ] the following condition hold:

$$\breve{w}_3 < \frac{a_6 a_8}{a_7 - a_8} \; . \qquad (3\,e)$$

On the other hand by using Routh-Hawirtiz criterion equation in [5] has roots (eigenvalues) with negative real parts if and only if

$R_1 > 0$, $R_3 > 0$ *and* $\Delta = R_1R_2 - R_3 > 0$

Straightforword computation shows that $\Delta > 0$

$$\text{provided that} \frac{a_3}{a_4(1-m)} > \frac{1}{2} \qquad (3\,f)$$

Now it is easy to verify that $R_1 > 0$ and $R_3 > 0$ under condition(2.4 g).Then all the eigenvalues λ_{w_1} , λ_{w_2} and λ_{w_3} of equation(2.9 d)have negative real parts. So, E_2 is locally

4. Finally, the Positive (Coexistence)Equilibrium point:

$E_3 = (w_1^*, w_2^*, w_3^*, w_4^*)$ *exists and it is locally asymptotically stable, as shown in [ ].*

IV. The local bifurcation analysis of system(2)

In this section, the effect of varying the parameter values on the dynamical behavior of the system (2) around each equilibrium points is studied. Recall that the existence of non hyperbolic equilibrium point of system(2) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorems an application to the Sotomayor's theorem for local bifurcation is appropriate.

Now, according to Jacobian matrix of system(2)given in equation(2.7), it is clear to verify that for any nonzero vector $V = (v_1, v_2, v_3, v_4)^T$ we have:

$$D^2 f(V,V) = \begin{pmatrix} -2\, v_2^2 \\ -2(1-m)v_2 v_3 \\ 2\, a_4(1-m)v_2 v_3 + 2\frac{a_5 a_6 v_3}{(a_6+w_3)^2}\left[\frac{v_3 w_4}{a_6+w_3} - v_4\right] \\ 2\frac{a_6 a_7 v_3}{(a_6+w_3)^2}\left[v_4 - \frac{v_3 w_4}{a_6+w_3}\right] \end{pmatrix} \quad (3.1)$$

$$\text{and} D^3 f(V,V,V) = \begin{pmatrix} 0 \\ 0 \\ -2\frac{a_5 a_6 v_3^2}{(a_6+w_3)^3}\left[\frac{3v_3 w_4}{a_6+w_3} + v_4\right] \\ 6\frac{a_6 a_7 v_3^2}{(a_6+w_3)^3}\left[\frac{v_3 w_4}{a_6+w_3} - v_4\right] \end{pmatrix}.$$

In the following theorems the local bifurcation conditions near the equilibrium point are established.

Theorem (3.1): If the parameter a_2 passes through the value $a_2^* = 1$, then the vanishing equilibrium point E_0 transforms into non-hyperbolic equilibrium point and system (2.2) possesses a transcritical bifurcation but no saddle-node bifurcation, nor pitchfork bifurcation can occur at E_0.

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Proof: *According to the Jacobian matrix $J(E_0)$given by Eq. (2.8 a) the system (2) at the equilibrium point E_0 has zero eigenvalue (say $\lambda_{0x} = 0$) at $a_2 = a_2^*$, and the Jacobian matrix J_0 with $a_2 = a_2^*$ becomes:*

$$J_0^* = J(a_2 = a_2^*) = \begin{pmatrix} -a_1 & 1 & 0 & 0 \\ a_1 & -1 & 0 & 0 \\ 0 & 0 & -a_3 & 0 \\ 0 & 0 & 0 & -a_6 \end{pmatrix}$$

Now, let $V^{[0]} = \left(v_1^{[0]}, v_2^{[0]}, v_3^{[0]}, v_4^{[0]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{0x} = 0$ .Thus $\left(J_0^* - \lambda_{0w_1} I\right)V^{[0]} = 0$, which gives: $v_2^{[0]} = a_1 v_1^{[0]}$, $v_3^{[0]} = 0$, $v_4^{[0]} = 0$ and $v_1^{[0]}$ any nonzero real number.

Let $\Psi^{[0]} = \left(\psi_1^{[0]}, \psi_2^{[0]}, \psi_3^{[0]}, \psi_4^{[0]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{0w_1} = 0$ of the matrix J_0^{*T}. Then we have, $\left(J_0^{*T} - \lambda_{0w_1} I\right)\Psi^{[0]} = 0$. By solving this equation for $\Psi^{[0]}$ we obtain, $\Psi^{[0]} = \left(\psi_1^{[0]}, \psi_1^{[0]}, 0, 0\right)^T$, where $\psi_1^{[0]}$ any nonzero real number.

Now, consider:

$$\frac{\partial f}{\partial a_2} = f_{a_2}(W, a_2) = \left(\frac{\partial f_1}{\partial a_2}, \frac{\partial f_2}{\partial a_2}, \frac{\partial f_3}{\partial a_2}, \frac{\partial f_4}{\partial a_2}\right)^T = (0, -w_2, 0, 0)^T .$$

So, $f_{a_2}(E_0, a_2^*) = (0,0,0,0)^T$ and hence $\left(\Psi^{[0]}\right)^T f_{a_2}(E_0, a_2^*) = 0$.

Therefore, according to Sotomayor's theorem the saddle-node bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{a_2}(W, a_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $Df_{a_2}(W, a_2)$ represents the derivative of $f_{a_2}(W, a_2)$with respect to $W = (w_1, w_2, w_3, w_4)^T$. Further ,it is observed that

$$Df_{a_2}(E_0, a_2^*)V^{[0]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1^{[0]} \\ a_1 v_1^{[0]} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -a_1 v_1^{[0]} \\ 0 \\ 0 \end{pmatrix},$$

$$\left(\Psi^{[0]}\right)^T\left[Df_{a_2}(E_0, a_2^*)V^{[0]}\right] = -a_1 v_1^{[0]}\psi_1^{[0]} \neq 0 .$$

Now, by substituting $V^{[0]}$ in (3.1) we get:

$$D^2 f(E_0, a_2^*)\left(V^{[0]}, V^{[0]}\right) = \begin{pmatrix} -2\, a_1^2\left(v_1^{[0]}\right)^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, it is obtain that:

$$\left(\Psi^{[0]}\right)^T D^2 f(E_0, a_2^*)\left(V^{[0]}, V^{[0]}\right) = -2\, a_1^2 \psi_1^{[0]}\left(v_1^{[0]}\right)^2 \neq 0 .$$

Thus, according to Sotomayor's theorem (1.15) system (2) has transcrirtical bifurcation at E_0 with the parameter $a_2 = a_2^*$.

Theorem;If the parameter a_3 passes through the value $a_3^* = a_4(1-m)\overline{w}_2$, then the free predatorsequilibrium point $E_1 = (\overline{w}_1, \overline{w}_2, 0, 0)$ transforms into non-hyperbolic equilibrium point and system (2) possesses a transcritical bifurcation but no saddle-node bifurcation, nor pitchfork bifurcation can occur at $E_1 = (\overline{w}_1, \overline{w}_2, 0, 0)$.

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Proof: *According to the Jacobian matrix J_1 given by Eq. (2.9 a) the system (2) at the equilibrium point E_1 has zero eigenvalue (say $\lambda_{1w_3} = 0$) at $a_3 = a_3^*$, and the Jacobian matrix J_1 with $a_3 = a_3^*$ becomes:*

$$J_1^* = J(a_3 = a_3^*) = \begin{pmatrix} -a_1 & 1 - 2\overline{w}_2 & 0 & 0 \\ a_1 & -a_2 & -(1-m)\overline{w}_2 & -(1-m)\overline{w}_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_6 + a_7(1-m)\overline{w}_2 \end{pmatrix}$$

Let $V^{[1]} = \left(v_1^{[1]}, v_2^{[1]}, v_3^{[1]}, v_4^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{1w_3} = 0$. Thus $\left(J_1^* - \lambda_{1w_3} I\right)V^{[1]} = 0$,which gives:

$$v_1^{[1]} = \frac{(1-2a_2)(1-m)}{a_1} v_3^{[1]}, \quad v_2^{[1]} = -(1-m)v_3^{[1]} \quad \text{and} \quad v_4^{[1]} = 0 ,$$

where $v_3^{[1]}$ any nonzero real number.

Let $\Psi^{[1]}=\left(\psi_1^{[1]},\psi_2^{[1]},\psi_3^{[1]},\psi_4^{[1]}\right)^T$ *be the eigenvector associated with the eigenvalue* $\lambda_{1w_3}=0$ *of the matrix* J_1^{*T}. *Then we have,* $\left(J_1^{*T}-\lambda_{1w_3}I\right)\Psi^{[1]}=0$. *By solving this equation for* $\Psi^{[1]}$ *we obtain,* $\Psi^{[1]}=\left(0,0,\psi_3^{[1]},0\right)^T$, *where* $\psi_3^{[1]}$ *any nonzero real number. Now, consider:*

$$\frac{\partial f}{\partial a_3}=f_{a_3}(W,a_3)=\left(\frac{\partial f_1}{\partial a_3},\frac{\partial f_2}{\partial a_3},\frac{\partial f_3}{\partial a_3},\frac{\partial f_4}{\partial a_3}\right)^T=(0,0,-w_3,0)^T\ .$$

So, $f_{a_3}(E_1,a_3^*)=(0,0,0,0)^T$ *and hence* $\left(\Psi^{[1]}\right)^T f_{a_3}(E_1,a_3^*)=0$.

Therefore, according to Sotomayor's theorem (1.15) the saddle-node bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{a_3}(W,a_3)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix},$$

where $Df_{a_3}(W,a_3)$ *represents the derivative of* $f_{a_3}(W,a_3)$ *with respect to* $W=(w_1,w_2,w_3,w_4)^T$ *.Further ,it is observed that*

$$Df_{a_3}(E_1,a_3^*)V^{[1]}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}v_1^{[1]}\\v_2^{[1]}\\v_3^{[1]}\\v_4^{[1]}\end{pmatrix}=\begin{pmatrix}0\\0\\-v_3^{[1]}\\0\end{pmatrix},$$

$$\left(\Psi^{[1]}\right)^T\left[Df_{a_3}(E_1,a_3^*)V^{[1]}\right]=-v_3^{[1]}\psi_3^{[1]}\neq 0\ .$$

Now, by substituting $V^{[1]}$ *in (3.1) we get:*

$$D^2f(E_1,a_3^*)\left(V^{[1]},V^{[1]}\right)=\begin{pmatrix}2(1-m)^2\left(v_3^{[1]}\right)^2\\2(1-m)^2\left(v_3^{[1]}\right)^2\\-2a_4(1-m)^2\left(v_3^{[1]}\right)^2\\0\end{pmatrix}.$$

Hence, it is obtain that:

$$\left(\Psi^{[1]}\right)^T D^2f(E_1,a_3^*)\left(V^{[1]},V^{[1]}\right)=-2a_4(1-m)^2\,\psi_3^{[1]}\left(v_3^{[1]}\right)^2\neq 0\ .$$

Thus, according to Sotomayor's theorem (1.15) system (2) has transcrirtical bifurcation at E_1 *with the parameter* $a_3=a_3^*$

Theorem (3.3): If the parameter a_8 passes through the value $a_8^*=\frac{a_7\breve{w}_4}{a_6+\breve{w}_3}$, then the free top predators equilibrium point $E_2=(\breve{w}_1,\breve{w}_2,\breve{w}_3,0)$ transforms into non-hyperbolic equilibrium point and system (2.2) possesses a transcritical bifurcation but no saddle-node bifurcation, nor pitchfork bifurcation can occur at $E_2=(\breve{w}_1,\breve{w}_2,\breve{w}_3,0)$.

Proof: *According to the Jacobian matrix* J_2 *given by Eq.* (2.10 a) *the system* (2.2) *at the equilibrium point* E_2 *has zero eigenvalue (say* $\lambda_{2w_4}=0$*) at* $a_8=a_8^*$, *and the Jacobian matrix* J_2 *with* $a_8=a_8^*$ *becomes:*

$J_2^*=J(a_8=a_8^*)=\left[e_{ij}\right]_{4\times4}$ *where*

$$n_{11}=-a_1,\ n_{12}=1-2\breve{w}_2,\ n_{13}=0,\ n_{14}=0,\ n_{21}=a_1,$$
$$n_{22}=-a_2-(1-m)\breve{w}_3,\ n_{23}=-(1-m)\breve{w}_2,\ n_{24}=0,$$
$$n_{31}=0,\ n_{32}=a_4(1-m)\breve{w}_3,\ n_{33}=-a_3+a_4(1-m)\breve{w}_2,$$
$$n_{34}=-\frac{a_5\breve{w}_3}{a_6+\breve{w}_3},\ n_{41}=0,\ n_{42}=0,\ n_{43}=0,\ n_{44}=0\ .$$

Let $V^{[2]}=\left(v_1^{[2]},v_2^{[2]},v_3^{[2]},v_4^{[2]}\right)^T$ *be the eigenvector corresponding to the eigenvalue* $\lambda_{2w_3}=0$ *.Thus* $\left(J_2^*-\lambda_{2w_3}I\right)V^{[2]}=0$ *,which gives:*

$$v_1^{[2]}=\frac{a_5[a_4(1-m)-2a_3]}{a_1a_6a_4^2(1-m)^2(1-a_2)-a_1a_3a_4}v_4^{[2]},$$

$$v_2^{[2]}=\frac{a_5(1-m)}{a_4a_6(1-m)+[a_4(1-m)-a_3]}v_4^{[2]}$$

and $v_3^{[2]}=-\frac{a_5}{a_4a_6(1-m)+[a_4(1-m)-a_3]}v_4^{[2]}$,

where $v_4^{[2]}$ *any nonzero real number .*

Let $\Psi^{[2]}=\left(\psi_1^{[2]},\psi_2^{[2]},\psi_3^{[2]},\psi_4^{[2]}\right)^T$ *be the eigenvector associated with the eigenvalue* $\lambda_{2w_3}=0$ *of the matrix* J_2^{*T} *.Then we have,* $\left(J_2^{*T}-\lambda_{2w_4}I\right)\Psi^{[2]}=0$. *By solving this equation for* $\Psi^{[2]}$ *we obtain,* $\Psi^{[2]}=\left(0,0,0,\psi_4^{[2]}\right)^T$, *where* $\psi_4^{[2]}$ *any nonzero real number. Now, consider:*

$$\frac{\partial f}{\partial a_8}=f_{a_8}(W,a_8)=\left(\frac{\partial f_1}{\partial a_8},\frac{\partial f_2}{\partial a_8},\frac{\partial f_3}{\partial a_8},\frac{\partial f_4}{\partial a_8}\right)^T=(0,0,0,-w_4)^T\ .$$

So, $f_{a_8}(E_2,a_8^*)=(0,0,0,0)^T$ *and hence* $\left(\Psi^{[2]}\right)^T f_{a_8}(E_2,a_8^*)=0$.

Therefore, according to Sotomayor's theorem (1.15) the saddle-node bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{a_8^*}(X,a_8)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix},$$

where $Df_{a_8}(w,a_8)$ *represents the derivative of* $f_{a_8}(X,a_8)$ *with respect to* $w=(w_1,w_2,w_3,w_4)^T$ *. Further, it is observed that*

$$Df_{a_8}(E_2\,,a_8^*)V^{[2]}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}v_1^{[2]}\\v_2^{[2]}\\v_3^{[2]}\\v_4^{[2]}\end{pmatrix}$$

$$=\begin{pmatrix}0\\0\\0\\-v_4^{[2]}\end{pmatrix},$$

$$\left(\Psi^{[2]}\right)^T\left[Df_{a_8}(E_2\,,a_8^*)V^{[2]}\right]=-v_4^{[2]}\psi_4^{[2]}\neq 0\ .$$

Now, by substituting $V^{[2]}$ *in (3.1) we get:*

$$D^2f(E_2\,,a_8^*)\left(V^{[2]}\,,V^{[2]}\right)$$

$$=\begin{pmatrix}-2\left(v_2^{[2]}\right)^2\\ -2(1-m)v_2^{[2]}v_3^{[2]}\\ 2v_3^{[2]}\left[a_4(1-m)v_2^{[2]}+\dfrac{a_6a_7v_4^{[2]}}{(a_6+w_3)^2}\right]\\ 2\dfrac{a_6a_7v_3^{[2]}v_4^{[2]}}{(a_6+\breve{w}_3)^2}\end{pmatrix}.$$

Hence, it is obtain that:

$$\left(\Psi^{[2]}\right)^TD^2f(E_2\,,a_8^*)\left(V^{[2]}\,,V^{[2]}\right)=2\frac{a_6a_7v_3^{[2]}v_4^{[2]}}{(a_6+\breve{w}_3)^2}\psi_4^{[2]}\neq 0\ .$$

Thus, according to Sotomayor's theorem (1.15) system (2.2) *has transcrirtical bifurcation at* E_2 *with the parameter* $a_8=a_8^*$.

Theorem (3.5): Suppose that the following conditions

$$a_7>a_8\,(1+(1-m)a_6\,)\qquad(3.4)$$

are satisfied. Then for the parameter value

$$a_2^*=1-(1-m)\frac{a_6a_8}{a_7-a_8}$$

system (2.2) at the equilibrium point $E_3=(w_1^*\,,w_2^*\,,w_3^*\,,w_4^*)$ has saddle-node bifurcation, but transcrirtical bifurcation, nor pitchfork bifurcation can occur at E_3.

Proof: *The characteristic equation given by Eq.*$(2.12\,b\,)$*having zero eigenvalue (say* $\lambda_4=0$ *) if and only if* $C_4=0$ *and then* E_3 *becomes a nonhyperbolic equilibrium point. Clearly the Jacobian matrix of system*(2.2)*at the equilibrium point* E_4 *with parameter* $a_2=a_2^*$ *becomes:* $J_4^*=J(a_2=a_2^*)=\left[\varepsilon_{ij}\right]_{4\times4}$ *where*

$\varepsilon_{ij}=d_{ij}$ *for all* $i\,,j=1,2,3,4$ *except* ε_{ij} *which is given by:*

$$\varepsilon_{22}=-a_2^*-(1-m)\frac{a_6a_7}{a_7-a_8}\ .$$

Note that, $a_2^*>0$ *provided that condition (3.4) holds.*

Let $V^{[3]}=\left(v_1^{[3]}\,,v_2^{[3]}\,,v_3^{[3]}\,,v_4^{[3]}\right)^T$ *be the eigenvector corresponding to the eigenvalue* $\lambda_4=0$ *. Thus* $(J_4^*-\lambda_4I)\,V^{[3]}=0$ *, which gives:*

$$v_1^{[3]}=\frac{1-2\,w_2^*}{a_1}v_2^{[3]}\,,\ v_3^{[3]}=\frac{(d_{12}+d_{22})}{(1-m)w_2^*}v_2^{[3]},$$

$$v_4^{[3]}=\frac{-(d_{12}+d_{22})a_6a_7w_4^*}{d_{23}(a_6+w_3^*)(a_6a_8+(a_8-a_7)w_3^*)}v_2^{[3]},$$

and $v_2^{[3]}$ *any nonzero real number. Here* $v_1^{[3]},v_4^{[3]}$ *not equal zero under the existence condition* $(2.5\,i\,)$.

Let $\Psi^{[3]}=\left(\psi_1^{[3]}\,,\psi_2^{[3]}\,,\psi_3^{[3]}\,,\psi_4^{[3]}\right)^T$ *be the eigenvector associated with the eigenvalue* $\lambda_4=0$ *of the matrix* J_3^{*T}*.Then we have,* $(J_3^{*T}-\lambda_4I)\Psi^3=0$. *By solving this equation for* $\Psi^{[3]}$ *we obtain:*

$$\Psi^{[3]}=\begin{pmatrix}\psi_2^{[3]}\\ \psi_2^{[3]}\\ -\dfrac{(d_{12}+d_{22})}{a_4(1-m)w_3^*}\psi_2^{[3]}\\ \dfrac{(d_{12}+d_{22})}{a_4(1-m)(a_6a_8+(a_8-a_7)w_3^*)}\psi_2^{[3]}\end{pmatrix},$$

where $\psi_2^{[4]}$ *any nonzero real number.*

Now,

$\frac{\partial f}{\partial a_2}=f_{a_2}(W\,,a_2)=\left(\frac{\partial f_1}{\partial a_2},\frac{\partial f_2}{\partial a_2},\frac{\partial f_3}{\partial a_2},\frac{\partial f_4}{\partial a_2}\right)^T=(0\,,-w_2\,,0\,,0)^T$. *So,* $f_{a_2}(E_3\,,a_2^*)=(0\,,-w_2^*\,,0\,,0)^T$ *and hence*

$$\left(\Psi^{[3]}\right)^Tf_{a_2}(E_3\,,a_2^*)=-w_2^*\psi_2^{[4]}\neq 0\ .$$

Therefore, according to Sotomayor's theorem (1.15) the saddle-node bifurcation occure at E_3 *under the existence condition (2.5 q).*

The Hopf bifurcation analysis of system $(\mathbf{2})$

In this section, the possibility of occurrence of a Hopf bifurcation near the postive equilibrium point of the system(2) *is investigated as shown in the following theorems.*

Theorem (3.8): *Suppose that the following conditions are satisfied:*

$max\left\{-l_6\,,\frac{-(l_3+d_{23}d_{32})}{d_{33}}\right\}<a_1<min\ F$ *where*

$F=\left\{d_{33}\,,\frac{l_3}{3(1-m)w_3^*}\,,\frac{-\Gamma_3\,l_2}{d_{34}d_{43}}\,,\frac{a_5a_6a_7w_3^*w_4^*}{d_{33}(a_6+w_3^*)^3}\,,\frac{d_{33}(-a_1d_{12}+d_{33})}{d_{33}+l_3}\right\}$

max $\{l_3\,,a_1\,l_3\,,2d_{33}l_4+3d_{11}\}<\frac{a_5a_6a_7w_3^*w_4^*}{(a_6+w_3^*)^3}<\frac{-a_1d_{12}}{2}$ *(3.8)*

$$C_3<\min\left\{\frac{C_1C_2}{2},\frac{C_1^2}{l_2}\right\}\qquad(3.9)$$

$$C_1^3-4\,\Delta_1>0\qquad(3.10)$$

Then at the parameter value $a_2=\bar{a}_2$ *, the system* (2.2) *has a Hopf bifurcation near the point* E_3.

Proof: *According to the Hopf bifurcation theorem (1.17), for n=4, the Hopf bifurcation can occur provided that* $C_i(\bar{a}_2)>0$ *;*

$i=1\,,3$ *,* $\Delta_1>0$ *,*

$C_1^3 - 4\,\Delta_1 > 0$ *and* $\Delta_2(\bar{a}_2) = 0$. *Straight forward computation gives that: if the following conditions* (2.10 *d*) , (2.10 *e*) *,(3.7),(3.8),(3.9) and (3.10) are hold, then* $C_i(\bar{a}_2) > 0$; $i = 1,3$, $\Delta_1 > 0$ *and* $C_1^3 - 4\,\Delta_1 > 0$.

Now, to verify the necessary and sufficient conditions for a Hopf bifurcation to occur we need to find a parameter satisfy $\Delta_2 = 0$. *Therefore, it is observed that* $\Delta_2 = 0$ *gives that:*

$$C_3(C_1C_2 - C_3) - C_1^2\,C_4 = 0$$

Now, by using Descartes rule Eq.(3.11) has a unique positive root say $a_2 = \bar{a}_2$ *such that:*

$$\mathcal{L}_1\bar{a}_2^{\,3} + \mathcal{L}_2\bar{a}_2^{\,2} + \mathcal{L}_3\bar{a}_2 + \mathcal{L}_4 = 0$$

$$\mathcal{L}_1 = l_1 l_2 < 0$$

$$\mathcal{L}_2 = (1-m)w_3^*(l_1 - d_{33})l_2 - d_{34}d_{43}(d_{34}d_{43} + 3d_{11} + 2l_1 d_{33}) + \Gamma_3 l_2 + d_{11}(d_{11}l_3 - d_{12}d_{21}d_{33} - d_{33}^2 l_1) - d_{12}d_{21}d_{33}l_1.$$

$$\mathcal{L}_3 = -(1-m)^2 w_3^{*2} l_1 l_2 + (1-m)w_3^*\big(d_{34}d_{43}d_{11}(3 + l_1)\big) + 2d_{34}d_{43}(d_{34}d_{43} - d_{33}l_2) - 2\Gamma_3 l_2 - 2d_{11}^2 l_3 + d_{12}d_{21}d_{33}l_1 - \Gamma_3 d_{11}(l_3 + l_2) + d_{12}d_{21}d_{33}(l_5 + 3(1-m)w_3^*) + d_{33}l_4(2d_{11}^2 - \Gamma_3 + 2(1-m)w_3^* d_{11})$$

$$\mathcal{L}_4 = (1-m)w_3^*\Gamma_3 l_1(l_2 + d_{12}d_{21}) - (1-m)^3 w_3^{*3} l_1 l_2 + \Gamma_3 d_{12}d_{21}(d_{11}^2 + d_{33}) + d_{33}d_{34}d_{43}(d_{12}d_{21} - d_{11}^2) + (1-m)w_3^* d_{33}^2(l_2 + l_4 - l_1^2 d_{11}) - (1-m)^2 w_3^{*2}\big(d_{11}(l_5 - d_{33}d_{12}d_{21} - d_{34}d_{43}) - d_{34}d_{43}(d_{34}d_{43} + d_{11}l_1)\big) - \Gamma_3 d_{11}(d_{34}d_{43}(l_1 - d_{11}l_3) + d_{11}) + d_{11}^2\Gamma_3 d_{23}d_{32} - d_{11}d_{33}(\Gamma_3 l_6 - d_{33}d_{12}d_{21}l_1 - \Gamma_3 d_{11}) + (1-m)w_3^* d_{11}d_{34}d_{43}(l_3 + d_{34}d_{43}).$$

and $l_1 = d_{11} + d_{33} > 0,$

$$l_2 = d_{34}d_{43} - d_{11}d_{33} > 0$$

$l_3 = d_{33}^2 - d_{23}d_{32} > 0,\quad l_4 = d_{34}d_{43} + d_{12}d_{21} < 0$

$l_5 = d_{33}^2 - d_{12}d_{21} > 0,\quad l_6 = d_{12}d_{21} + d_{23}d_{32} < 0$

Now, at $a_2 = \bar{a}_2$ *the characteristic equation can be written as:*

$$\left(\lambda_4^2 + \frac{C_3}{C_1}\right)\left(\lambda_4^2 + C_1\lambda_4 + \frac{\Delta_1}{C_1}\right) = 0 \quad ,$$

which has four roots

$$\lambda_{41,2} = \pm i\sqrt{\frac{C_3}{C_1}} \text{ and } \lambda_{4\,3,4} = \frac{1}{2}\left(-C_1 \pm \sqrt{C_1^2 - 4\frac{\Delta_1}{C_1}}\right).$$

Clearly, at $a_2 = \bar{a}_2$ *there are two pure imaginary eigenvalues (* $\lambda_{4\,1}$ *and* $\lambda_{4\,2}$ *) and two eigenvalues which are real and negative. Now for all values of* a_2 *in the neighborhood of* $\bar{a}_2$*, the roots in general of the following form:*

$$\lambda_{4\,1} = \alpha_1 + i\alpha_2 \ ,\ \lambda_{4\,2} = \alpha_1 - i\alpha_2$$

$$,\ \lambda_{4\,3,4} = \frac{1}{2}\left(-C_1 \pm \sqrt{C_1^2 - 4\frac{\Delta_1}{C_1}}\right).$$

Clearly, $Re\big(\lambda_{4k}(a_2)\big)\big|_{a_2=\bar{a}_2} = \alpha_1(\bar{a}_2) = 0$, $k = 1,2$ *that means the first condition of the necessary and sufficient conditions for Hopf bifurcation is satisfied at* $a_2 = \bar{a}_2$*. Now, according to verify the transversality condition we must prove that:*

$$\bar{\Theta}(\bar{a}_2)\bar{\Psi}(\bar{a}_2) + \bar{\Gamma}(\bar{a}_2)\bar{\Phi}(\bar{a}_2) \neq 0 \quad ,$$

where $\bar{\Theta}$, $\bar{\Psi}$, $\bar{\Gamma}$ *and* $\bar{\Phi}$ *are given in (1.29). Note that for* $a_2 = \bar{a}_2$ *we have* $\alpha_1 = 0$ *and* $\alpha_2 = \sqrt{\frac{C_3}{C_1}}$ *, substituting into (1.29) gives the following simplifications:*

$$\bar{\Psi}(\bar{a}_2) = -2\,C_3(\bar{a}_2)\ ,$$

$$\bar{\Phi}(\bar{a}_2) = 2\,\frac{\alpha_2(\bar{a}_2)}{C_1}(C_1C_2 - 2\,C_3)\ ,$$

$$\bar{\Theta}(\bar{a}_2) = C_4'(\bar{a}_2) - \frac{C_3}{C_1}C_2'(\bar{a}_2)\ ,$$

$$\bar{\Gamma}(\bar{a}_2) = \alpha_2(\bar{a}_2)\left(C_3'(\bar{a}_2) - \frac{C_3}{C_1}C_1'(\bar{a}_2)\right)\ ,$$

where

$$C_1' = \frac{dC_1}{da_2}\Big|_{a_2=\bar{a}_2} = 1\ ,$$

$$C_2' = \frac{dC_2}{da_2}\Big|_{a_2=\bar{a}_2} = -f_1\ ,$$

$$C_3' = \frac{dC_3}{da_2}\Big|_{a_2=\bar{a}_2} = -f_2\ ,$$

$$C_4' = \frac{dC_4}{da_2}\Big|_{a_2=\bar{a}_2} = d_{11}d_{34}d_{43}\ .$$

Then by using Eq.(1.30) we get that:

$\bar{\Theta}(\bar{a}_2)\bar{\Psi}(\bar{a}_2) + \bar{\Gamma}(\bar{a}_2)\bar{\Phi}(\bar{a}_2) = -2C_3(\bar{a}_2)\left(d_{11}d_{34}d_{43} + \frac{C_3 l_2}{C_1}\right) - 2\,\frac{\alpha_2(\bar{a}_2)^2}{C_1}(C_1C_2 -$ *2 C3(l2+C3C1)* .

Now, according to conditions (3.8) and (3.9) we have:

$$\bar{\Theta}(\bar{a}_2)\bar{\Psi}(\bar{a}_2) + \bar{\Gamma}(\bar{a}_2)\bar{\Phi}(\bar{a}_2) \neq 0\ .$$

So, we obtain that the Hopf bifurcation occurs around the equilibrium point E_3 *at the parameter* $a_2 = \bar{a}_2$.

■

V. Numerical analysis of system(2)

In this section, the dynamical behavior of system (2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the

positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in Fig.(1).

$$a_1 = 0.7\ , a_2 = 0.2\ , a_3 = 0.1\ , a_4 = 0.5\ , a_5 = 0.6, m = 0.8\ \ (14)$$

$$a_6 = 0.4\ , a_7 = 0.5\ , a_8 = 0.2\ , m = 0.8\ \ (14)$$

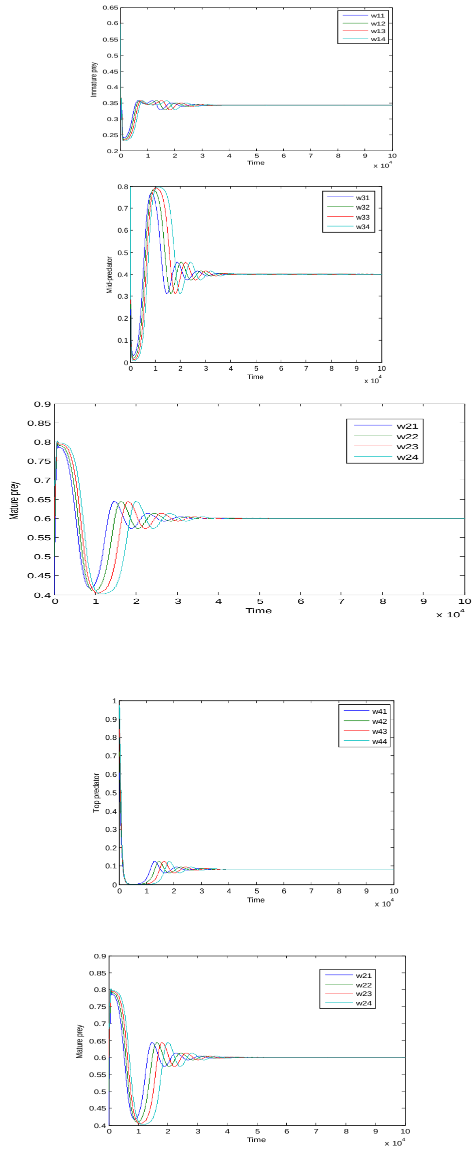

Fig.(1): Time series of the solution of system(2)that started from four different initialpoints(0.3 , 0.4 , 0.5 , 0.6), (2.5,0.5 , 2.5 , 1.5), (2 , 1.5 , 2.5,2.5) and (2.5 , 2.5,1.5 , 0.5) for the data given by(2.1). (a) trajectories of w_1 as a function of time, (b) trajectories of w_2 as a function of time, (c) trajectories of w_3 as a function of time, (d) trajectories of w_4 as a function of time .

Clearly, Fig. (1) shows that system (2) has a globally asymptotically stable as the solution of system(2)approaches asymptotically to the positive equilibrium point $E_3 = (0.34 , 0.6 , 0.4 , 0.08)$ starting from four different initial points and this is confirming our obtained analytical results.

Now, in order to discuss the effect of the parameters values of system(2)on the dynamical behavior of the system, the system is solved numerically for the data given in(14)with varying one parameter at each time. It is observed that for the data given in(14)with $0.1 \le a_1 < 1$, the solution of system(2)approaches asymptotically to the positive equilibrium point as shown in Fig.(2)for typical value $a_1 = 0.2$.

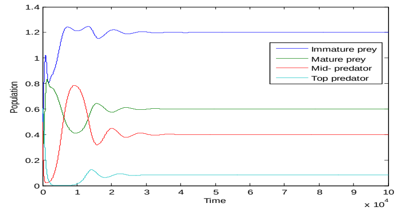

Fig.(2): Time series of the solution of system (2) for the data given by(14) with $a_1 = 0.2$, which approaches to (1.2 ,0.6 ,0.4 ,0.08) in the interior of R^4_+.

By varying the parameter a_2 and keeping the rest of parameters values as in (14) , it is observed that for $0.1 \le a_2 < 0.44$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_3 , while for $0.44 \le a_2 < 0.6$ the solution of system (2) approaches asymptotically to $E_2 = (\breve{w}_1 , \breve{w}_2 , \breve{w}_3 , 0)$ in the interior of the positive quadrant of $w_1 w_2 w_3 -$ plane as shownin Fig. (3) for typical value $a_2 = 0.5$.

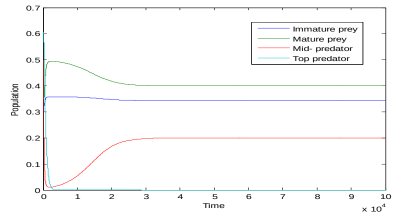

Fig.(3): Times series of the solution of system (2)for the data given by (14)with $a_2 = 0.5$. which approaches to(0.34 , 0.4 , 0.2 , 0) in the interior of the positive quadrant of $w_1 w_2 w_3 -$ plane .

while for $0.6 \le a_2 < 1$ the solution of system (2) approaches asymptotically to $E_1 = (\overline{w}_1 , \overline{w}_2 , 0 , 0)$ in the interior of the positive quadrant of $w_1 w_2 -$ plane as shown in Fig. (4)for typical value $a_2 = 0.8$.

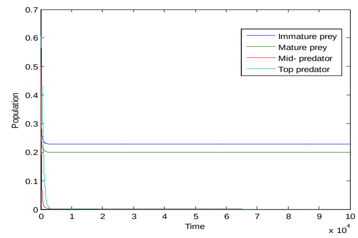

Fig.(4):Times series of the solution of system (2) for the data given by (14) with $a_2 = 0.8$ which

approaches to $(0.23, 0.2, 0, 0)$ *in the interior of the positive quadrant of* $w_1w_2 - plane$.

On the other hand varying the parameter a_3 *and keeping the rest of parameters values as in* (14), *it is observed that for* $0.1 \leq a_3 < 0.145$ *the solution of system* (2) *approaches asymptotically to the positive equilibrium point* E_3. *while for* $0.145 \leq a_3 < 0.2$ *the solution of system* (2) *approaches asymptotically to* $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ *in the interior of the positive quadrant of* $w_1w_2w_3 - plane$. *for* $0.2 \leq a_3 < 1$ *the solution of system* (2) *approaches asymptotically to* $E_1 = (\overline{w}_1, \overline{w}_2, 0, 0)$ *in the interior of the positive quadrant of* $w_1w_2 - plane$.

Moreover, varying the parameter a_4 *and keeping the rest of parameters values as in* (14), *it is observed that for* $0.1 \leq a_4 < 0.26$ *the solution of system* (2) *approaches asymptotically to* $E_1 = (\overline{w}_1, \overline{w}_2, 0, 0)$ *in the interior of the positive quadrant of* $w_1w_2 - plane$ *, while for* $0.26 < a_4 \leq 0.4$ *the solution of system* (2) *approaches asymptotically to* $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ *in the interior of the positive quadrant of* $w_1w_2w_3 - plan$. *and for the* $0.4 < a_4 < 1$ *the solution of system* (2) *approaches asymptotically to the positive equilibrium point* E_3.

For the parameter $0.5 < a_5 \leq 1$ *the solution of system* (2) *approaches asymptotically to a positive equilibrium point* E_3.

For the parameter $0.1 \leq a_6 < 0.2$ *the solution of system* (2) *approaches asymptotically to* $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ *in the interior of the positive quadrant of* $w_1w_2w_3 - plan$. *While for the* $0.2 \leq a_6 < 0.5$ *the solution of system* (2) *approaches asymptotically to a positive equilibrium point* E_3.

For the parameters values given in (14) *with* $0.4 < a_7 < 0.6$ *the solution of system* (2) *approaches asymptotically to the positive equilibrium point* E_3.

For the parameter $0.1 \leq a_8 < 0.32$ *the solution of system* (2) *approaches asymptotically to the positive equilibrium point* E_3. *while for the* $0.32 \leq a_8 < 1$ *the solution of system* (2) *approaches asymptotically to* $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ *in the interior of the positive quadrant of* $w_1w_2w_3 - plan$.

Moreover, varying the parameter m *and keeping the rest of parameters values as in* (14), *it is observed that for* $0.1 \leq m < 0.7$ *the solution of system* (2) *approaches asymptotically to the positive equilibrium point* E_3. *while for* $0.7 \leq m < 0.75$ *the solution of system* (2) *approaches asymptotically to* $E_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ *in the interior of the positive quadrant of* $w_1w_2w_3 - plane$. *and for* $0.75 \leq m < 1$ *the solution of system* (2) *approaches asymptotically to* $E_1 = (\overline{w}_1, \overline{w}_2, 0, 0)$ *in the interior of the positive quadrant of* $w_1w_2 - plane$.

Finally, the dynamical behavior at the vanishing equilibrium point $E_0 = (0, 0, 0, 0)$ *is investigated by choosing* $a_2 = 2$ *and keeping other parameters fixed as given in* (14), *and then the solution of system* (2) *is drawn in Fig.* (5).

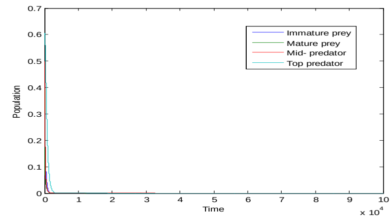

Fig. (5): *Times series of the solution of system* (2) *for the data given by* (14) *of with* $a_2 = 2$, *which approaches* $(0, 0, 0, 0)$.

VI. Conclusions and discussion:

In this chapter, we proposed and analyzed an ecological model that described the dynamical behavior of the food chain real system. The model included four non-linear autonomous differential equations that describe the dynamics of four different population, namely first immature prey (N_1), mature prey (N_2), mid-predator (N_3) and (N_4) which is represent the top predator. The boundedness of system (2.2) has been discussed. The existence condition of all possible equilibrium points are obtain. The local as well as global stability analyses of these points are carried out. Finally, numerical simulation is used to specific the control set of parameters that affect the dynamics of the system and confirm our obtained analytical results. Therefore system (2.2) has been solved numerically for different sets of initial points and different sets of parameters starting with the hypothertical set of data given by Eq.(4.1) and the following observations are obtained.

1. System (2) has only one type of attractor in Int. R_+^4 approaches to globally stable point.

2. For the set hypothetical parameters value given in (14), the system (2) approaches asymptotically to globally stable positive point $E_3 = (0.34, 0.6, 0.4, 0.08)$.Further, with varying one parameter each time, it is observed that varying the parameter values, $a_i, i = 1,5$ *and* 7 do not have any effect on the dynamical behavior of system (2) and the solution of the system still approaches to positive equilibrium point $E_3 = (w_1^*, w_2^*, w_3^*, w_4^*)$.

3. As the natural death rate of mature prey a_2 increasing to 0.43 keeping the rest of parameters as in eq.(14),the solution of system (2) approaches to positive equilibrium point E_3. However if $0.44 \leq a_2 < 0.6$, then the top

predator will face extinction then the trajectory transferred from positive equilibrium point to the equilibrium point $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ moreover, increasing $a_2 \geq 0.6$ will causes extinction in the mid-predator and top predator then the trajectory transferred from equilibrium point $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ to the equilibrium point $\mathrm{E}_1 = (\bar{w}_1, \bar{w}_2, 0, 0)$, thus, the a_2 parameter is bifurcation point.

3. As the natural death rate of top predator a_3 increasing to 0.144 keeping the rest of parameters as in eq.(14),the solution of system (2) approaches to positive equilibrium point E_3. However if $0.145 \leq a_3 < 0.2$, then the top predator will face extinction then the trajectory transferred from positive equilibrium point to the equilibrium point $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ moreover, increasing $0.2 \leq a_3$ will causes extinction in the mid-predator and top predator then the trajectory transferred from equilibrium point $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ to the equilibrium point $\mathrm{E}_1 = (\bar{w}_1, \bar{w}_2, 0, 0)$, thus, the a_3 parameter is bifurcation point.

4. As the natural death rate of top predator a_4 increasing to 0.1 keeping the rest of parameters as in eq.(14),the solution of system (2) approaches to $E_1 = (\bar{w}_1, \bar{w}_2, 0, 0)$. However if $0.26 \leq a_4 \leq 0.4$, then the trajectory transferred from $E_1 = (\bar{w}_1, \bar{w}_2, 0, 0)$ to the equilibrium point $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ moreover, increasing $a_4 > 0.4$ will then the trajectory transferred from equilibrium point $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ to the positive equilibrium point $E_3 = (w_1^*, w_2^*, w_3^*, w_4^*)$, thus, the a_4 parameter is bifurcation point.

5. As the half saturation rate of top predator a_6 decreases under specific value keeping the rest parameters as in eq(14) the top predator will face extinction and the solution of system (2) approaches to $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$. but where a_6 increases above specific value the trajectory transferred from $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ to the positive equilibrium point $E_3 = (w_1^*, w_2^*, w_3^*, w_4^*)$,Thus, the parameter a_6 is a bifurcation point.

6. As the natural death rateof the top predator $a_8 \geq 0.1$ increasing keeping the rest of parameters as in eq(14) system (2) has asymptotically stable positive point in Int.R_+^4.However increasing $a_8 \geq 0.32$ will causes extinction in the top predator then the trajectory transferred from positive equilibrium point to the $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$. Thus, the parameter a_8 is a bifurcation point.

7. As the number of prey inside the refuge m increasing to 0.68 keeping the rest of parameters as in eq.(14), the solution of system (2) approaches to positive equilibrium point E_3. However if $0.7 \leq \mathrm{m} < 0.75$ then the top predator will face extinction then the trajectory transferred from positive equilibrium point to the equilibrium point $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ moreover, increasing $0.75 \leq m$ will causes extinction in the mid-predator and top predator then the trajectory transferred from equilibrium point $\mathrm{E}_2 = (\breve{w}_1, \breve{w}_2, \breve{w}_3, 0)$ to the equilibrium point $\mathrm{E}_1 = (\bar{w}_1, \bar{w}_2, 0, 0)$, thus, the m parameter is bifurcation point.

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